

LOGARITHMIC HESSE'S PROBLEM

T. FASSARELLA

ABSTRACT. We show that if a Laurent polynomial on the coordinate ring of the complex algebraic torus on n variables has vanishing logarithmic Hessian, then up to an automorphism of the torus, the Laurent polynomial depends on at most $n - 1$ variables.

1. INTRODUCTION

Let $f \in \mathbb{C}[x_1, \dots, x_n]$, $n \geq 1$, be a polynomial and consider the *Hessian matrix*

$$Hf(x) := (f_{x_i x_j}(x))_{1 \leq i, j \leq n}.$$

The polynomial f has *vanishing Hessian* if the determinant $\det Hf \in \mathbb{C}[x_1, \dots, x_n]$ is null. This is equivalent to the condition on the derivatives of f being algebraically dependent, that is, f has vanishing Hessian if and only if the *affine polar map*

$$\begin{aligned} \nabla_f : \mathbb{C}^n &\longrightarrow \mathbb{C}^n \\ x &\longmapsto (f_{x_1}(x), \dots, f_{x_n}(x)) \end{aligned}$$

is not dominant.

Hesse claimed in [7, 8] that a homogeneous polynomial on n variables has vanishing Hessian if and only if after a suitable coordinate change, the polynomial depends on at most $n - 1$ variables. While the "if" implication is trivial, the "only if" statement is not trivial at all. The Hesse problem was taken up by Gordan–Noether in [5], who showed that the question has an affirmative answer for $n \leq 4$, but is false in general for $n \geq 5$. See [11, 12, 13, 10] for a review of the counterexamples constructed by Gordan–Noether. See also [1] for a modern overview of the known methods to deal with the problem. The case of nonhomogeneous polynomials was studied in [2] where a classification for small n was obtained.

In this paper we will consider polynomials $f \in S_n = \mathbb{C}[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}]$, that is, in the coordinate ring of the torus $(\mathbb{C}^*)^n$. An element $f \in S_n$ is called a *Laurent polynomial* on n variables. In this case we shall work with the *logarithmic polar map* associated to f defined as follows

$$\begin{aligned} L_f : (\mathbb{C}^*)^n &\longrightarrow \mathbb{C}^n \\ x &\longmapsto (x_1 f_{x_1}(x), \dots, x_n f_{x_n}(x)). \end{aligned}$$

As in the affine case, we note that L_f can be seen as the logarithmic Gauss map associated to the hypersurface $Z = \{f = 0\} \subset (\mathbb{C}^*)^n$, that is, the map that takes $x \in Z$ to the left translation to unity of the hyperplane $T_x Z \subset T_x(\mathbb{C}^*)^n$ (see example 1).

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We consider now the matrix

$$Af(x) := ((x_i \cdot f_{x_i})_{x_j}(x))_{i,j=1,\dots,n},$$

which will be called *logarithmic Hessian matrix* of f . We shall say that f has *vanishing logarithmic Hessian* if the determinant $\det Af \in S_n$ is null. If f is a Laurent polynomial, the condition of vanishing logarithmic Hessian is equivalent to the non-dominance hypothesis on L_f .

The main objective of this paper is to show that – as a counterpart to Hesse's problem – in the case of Laurent polynomials on the torus, the analogue of Hesse's claim is true. That is, if a Laurent polynomial on n variables has vanishing logarithmic Hessian then after an automorphism of the torus, the Laurent polynomial depends on at most $n - 1$ variables. This is Theorem 1. Before that we show that the fibers of L_f have a good behavior, more precisely, under the hypothesis of vanishing logarithmic Hessian, the logarithmic Gauss map associated to the foliation determined by its fibers is constant. This is Proposition 2.

2. FIBERS OF THE AFFINE POLAR MAP

In order to study the fibers of the logarithmic polar map we first restrict our attention to the fibers of the affine polar map. Let f be a holomorphic map defined on an open subset U of \mathbb{C}^n . Taking the derivatives of f we can define, as in the polynomial case, the affine polar map

$$\begin{aligned} \nabla_f : U &\longrightarrow \mathbb{C}^n \\ x &\longmapsto (f_{x_1}(x), \dots, f_{x_n}(x)). \end{aligned}$$

In the following proposition we use the same idea as in [4, Appendix] to study its fibers.

Proposition 1. *If the affine polar map $\nabla_f : U \longrightarrow \mathbb{C}^n$ has constant rank $n - k < n$, then a fiber of ∇_f is an union of open subsets of affine linear subspaces of dimension k of \mathbb{C}^n .*

Proof. We set $r = n - k$. Since the problem is local on U , one can take a parametrization $\varphi : S \times T \longrightarrow U$, where $S \subset \mathbb{C}^k$, $T \subset \mathbb{C}^r$ are open sets and $S \times \{t\}$ are fibers of $\Gamma := \nabla_f \circ \varphi$.

We have to show that $\varphi(S \times \{t\})$ is an open subset of some affine linear subspace on \mathbb{C}^n . Denote by $z = (s, t) \in S \times T$ and φ_{s_i} , φ_{t_j} , $i = 1, \dots, k$, $j = 1, \dots, r$, the natural derivatives. We will show that $\varphi(S \times \{t\}) \subset F_z$ where

$$F_z = \text{span}\{\varphi_{s_1}(z), \dots, \varphi_{s_k}(z)\} = \ker d\nabla_f(x).$$

We first note that $F_z = \tilde{F}_z$, where

$$\tilde{F}_z = \{V \in \mathbb{C}^n \mid \langle \Gamma_{t_j}(z), V \rangle = 0, j = 1, \dots, r\}.$$

Since $d\nabla_f(x)$ self-adjoint and $\Gamma_{t_j} = d\nabla_f \cdot \varphi_{t_j}$ we get

$$(1) \quad \langle \Gamma_{t_j}, \varphi_{s_i} \rangle = \langle \varphi_{t_j}, d\nabla_f \cdot \varphi_{s_i} \rangle = 0.$$

Hence $F_z \subset \tilde{F}_z$. But $\dim F_z = \dim \tilde{F}_z$, therefore $F_z = \tilde{F}_z$.

Now we claim that $\varphi_{s_i s_j}(z) \in \tilde{F}_z$. In fact, it follows from $\Gamma_{s_l} \equiv 0$ for all $l = 1, \dots, k$, and (1) that

$$\langle \Gamma_{t_l}, \varphi_{s_i s_j} \rangle = \langle \Gamma_{t_l}, \varphi_{s_i s_j} \rangle + \langle \Gamma_{t_l s_i}, \varphi_{s_j} \rangle = \langle \Gamma_{t_l}, \varphi_{s_j} \rangle_{s_i} = 0.$$

Therefore $\varphi_{s_i s_j}(z) \in F_z$, that is, φ satisfies the following system of PDE:

$$(2) \quad \varphi_{s_i s_j} = \sum_{l=1}^k \mu_{ijl} \cdot \varphi_{s_l},$$

for some holomorphic functions μ_{ijl} .

By a linear coordinate change, we may assume that $z = (0, 0)$ and

$$(3) \quad F_z = \{x \in \mathbb{C}^n \mid x_{k+1} = \dots = x_n = 0\}.$$

In order to show that $\varphi(S \times \{0\}) \subset F_z$, we have to prove that

$$\varphi_{s_i}^{k+1}(s, 0) = \dots = \varphi_{s_i}^n(s, 0) = 0 \quad \forall i = 1, \dots, k,$$

where $(\varphi_{s_i}^1(s, 0), \dots, \varphi_{s_i}^n(s, 0))$ are the components of $\varphi_{s_i}(s, 0)$. By (2) and (3), one obtains the following system of PDE:

$$\begin{pmatrix} \frac{d}{ds_j} \varphi_{s_1}^m \\ \vdots \\ \frac{d}{ds_j} \varphi_{s_k}^m \end{pmatrix} = (\mu_{ijl})_{1 \leq i, l \leq k} \cdot \begin{pmatrix} \varphi_{s_1}^m \\ \vdots \\ \varphi_{s_k}^m \end{pmatrix}, \quad \varphi_{s_i}^m(0, 0) = 0,$$

for each $j = 1, \dots, k$ and $m = k+1, \dots, n$.

Denoting by $\xi_i(s_j) = \varphi_{s_i}^m(0, \dots, 0, s_j, 0, \dots, 0)$ we obtain the following homogeneous system of ODE:

$$\begin{pmatrix} \xi_1' \\ \vdots \\ \xi_k' \end{pmatrix} = (\mu_{ijl})_{1 \leq i, l \leq k} \cdot \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_k \end{pmatrix}, \quad \xi_i(0) = 0.$$

This system has only the trivial solution by uniqueness. Therefore $\varphi_{s_i}^m(s, 0)$ vanish on all coordinate axes for every $i = 1, \dots, k$ and $m = k+1, \dots, n$. Since we can consider the initial condition at an arbitrary point of one of the coordinate axes, an induction argument on k shows that $\varphi_{s_i}^{k+1}(s, 0) = \dots = \varphi_{s_i}^n(s, 0) = 0$. \square

3. FIBERS OF THE LOGARITHMIC POLAR MAP

Let M be a complex variety and $1 \leq k < \dim(M)$. A *singular holomorphic foliation* \mathcal{F} of dimension k on M is determined by a line bundle \mathcal{L} and an element $\omega \in H^0(M, \Omega_M^{n-k} \otimes \mathcal{L})$ satisfying

- (i) $\text{codim}(\text{Sing}(\omega)) \geq 2$ where $\text{Sing}(\omega) = \{x \in M \mid \omega(x) = 0\}$;
- (ii) ω is integrable.

By definition ω is integrable if and only if for every point $x \in M \setminus \text{Sing}(\omega)$ there exist a neighborhood $V \subset M$ of x and 1-forms $\alpha_1, \dots, \alpha_{n-k} \in \Omega_M^1(V)$ such that

$$\omega|_V = \alpha_1 \wedge \dots \wedge \alpha_{n-k} \quad \text{and} \quad d\alpha_i \wedge \omega|_V = 0 \quad \forall i = 1, \dots, n-k.$$

The *singular set* of \mathcal{F} is by definition equal to $\text{Sing}(\omega)$. The integrability condition (ii) determines in an analytic neighborhood of every regular point x , i.e., $x \in M \setminus \text{Sing}(\omega)$ a holomorphic fibration with relative tangent sheaf coinciding with the subsheaf of TM determined by the kernel of ω . Analytic continuation of the fibers of this fibration describes the *leaves* of \mathcal{F} .

Let G be a complex algebraic Lie group and \mathfrak{g} its Lie algebra. Let \mathcal{F} be a singular holomorphic foliation of dimension k on G . We shall consider the left Gauss map associated to \mathcal{F} .

For each $x \in G$ let

$$\begin{aligned} l_x : G &\longrightarrow G \\ \tilde{x} &\longmapsto x\tilde{x} \end{aligned}$$

be the left translation by x and denote by $G(k, \mathfrak{g})$ the grassmannian of k -dimensional linear subspaces of \mathfrak{g} . For each $x \in G$ regular point of \mathcal{F} , we have a well defined k -dimensional linear subspace $T_x\mathcal{F} \subset T_xG$ which is the tangent space of the leaf of \mathcal{F} through x . Thus the *left Gauss map associated to \mathcal{F}* is the following rational map

$$\begin{aligned} \mathcal{G}(\mathcal{F}) : G &\dashrightarrow G(k, \mathfrak{g}) \\ x &\longmapsto d(l_{x^{-1}})(x)(T_x\mathcal{F}) \end{aligned}$$

defined only on the set of regular points of \mathcal{F} . This is similar to the left Gauss map defined by Kapranov in [9] for a hypersurface $Z \subset G$.

Example 1. Let $f \in S_n$ be a Laurent polynomial on n variables and $G = (\mathbb{C}^*)^n$. We consider its zero locus $Z = \{x \in (\mathbb{C}^*)^n \mid f(x) = 0\}$. The tangent space of Z at x is given by $T_xZ = \{V \in \mathbb{C}^n \mid f_{x_1}(x)(v_1 - x_1) + \cdots + f_{x_n}(x)(v_n - x_n) = 0\}$. Taking the left translation to unity of $(\mathbb{C}^*)^n$ one obtains

$$d(l_{x^{-1}})(x)(T_xZ) = \{W \in \mathbb{C}^n \mid x_1 f_{x_1}(x)(w_1 - 1) + \cdots + x_n f_{x_n}(x)(w_n - 1)\}.$$

Therefore $d(l_{x^{-1}})(x)(T_xZ) = (x_1 f_{x_1}(x) : \cdots : x_n f_{x_n}(x)) \in G(n-1, \mathfrak{g}) \cong \mathbb{P}^{n-1}$.

Let \mathcal{F} be the foliation of dimension $n-1$ on $(\mathbb{C}^*)^n$ which has as leaves, fibers of the regular function determined by f . With the above argument on each leaf we have that the left Gauss map of \mathcal{F} is just

$$\begin{aligned} \mathcal{G}(\mathcal{F}) : (\mathbb{C}^*)^n &\dashrightarrow \mathbb{P}^{n-1} \\ x &\longmapsto (x_1 f_{x_1}(x) : \cdots : x_n f_{x_n}(x)). \end{aligned}$$

From now on we consider the case $G = (\mathbb{C}^*)^n$. The left Gauss map for foliations on $(\mathbb{C}^*)^n$ will be called the *logarithmic Gauss map*. Let $f \in S_n$ be a Laurent polynomial on n variables. Suppose the logarithmic polar map L_f has rank $n-k$, $1 \leq k \leq n-1$. By the implicit function theorem, there is a singular holomorphic foliation \mathcal{F}_f on $(\mathbb{C}^*)^n$ of dimension k on which L_f is constant on each leaf. We shall refer to \mathcal{F}_f as the *foliation determined by fibers* of L_f . If L_f has rank $n-k=0$, that is L_f is constant, we still denote by $\mathcal{G}(\mathcal{F}_f)$ the constant map taking a point $x \in (\mathbb{C}^*)^n$ to the Lie algebra \mathfrak{g} . This odd remark will be useful in the proposition below.

Proposition 2. *Let $f \in S_n$ be a Laurent polynomial. Suppose f has vanishing logarithmic Hessian. Then the logarithmic Gauss map $\mathcal{G}(\mathcal{F}_f)$ is constant.*

Proof. We consider the holomorphic map $g := f \circ \mathcal{E}$, where \mathcal{E} is the following recovering of $(\mathbb{C}^*)^n$

$$\begin{aligned} \mathcal{E} : \mathbb{C}^n &\longrightarrow (\mathbb{C}^*)^n \\ y &\longmapsto (\exp(y_1), \dots, \exp(y_n)). \end{aligned}$$

We have a commutative diagram

$$\begin{array}{ccc} \mathbb{C}^n & & \\ \mathcal{E} \downarrow & \searrow \nabla_g & \\ (\mathbb{C}^*)^n & \xrightarrow{L_f} & \mathbb{C}^n \end{array}$$

where ∇_g is the affine polar map associated to g .

We set $x \in (\mathbb{C}^*)^n$ such that $dL_f(x)$ has maximal rank $n-k$, $1 \leq k \leq n$. If $k = n$, there is nothing to do. Let us suppose $1 \leq k \leq n-1$. Let E be an irreducible component of $L_f^{-1}(L_f(x))$. It follows from Proposition 1 that $E = \mathcal{E}(F)$ where F is an affine linear subspace of \mathbb{C}^n . Then E is a left coset of an irreducible algebraic subgroup in $(\mathbb{C}^*)^n$. But any irreducible algebraic subgroup H of $(\mathbb{C}^*)^n$ can be written in the form

$$H = H_\Lambda = \{x \in (\mathbb{C}^*)^n \mid x^\lambda = 1 \ \forall \lambda \in \Lambda\},$$

where Λ is a primitive lattice of \mathbb{Z}^n of rank $n-k$. So $E = x_0 \cdot H_\Lambda$ for some H_Λ . We notice that $H_\Lambda = \mathcal{E}(F_M)$, $F_M = M \otimes \mathbb{C}$ where M is the primitive lattice

$$M = \{(y_1, \dots, y_n) \in \mathbb{Z}^n \mid \lambda_1 y_1 + \dots + \lambda_n y_n = 0 \ \forall (\lambda_1, \dots, \lambda_n) \in \Lambda\}.$$

Let T_y be the translation by y in \mathbb{C}^n . By the equality $l_{x^{-1}} \circ \mathcal{E} = \mathcal{E} \circ T_{-y}$ one obtains that $d(l_{x^{-1}})(x)(T_x \mathcal{F}) = F_M$. In particular we can choose a basis of $d(l_{x^{-1}})(x)(T_x \mathcal{F})$ of vectors with integer coordinates. Then the image of $\mathcal{G}(\mathcal{F}_f)$ is contained in the subset of $G(k, \mathfrak{g}) = G(k-1, \mathbb{P}\mathfrak{g}) \subset \mathbb{P}^N$ on which the points can be chosen with integer coordinates. Since $\mathcal{G}(\mathcal{F}_f)$ is a rational map it must be constant. \square

Remark 1. We notice that the condition $\mathcal{G}(\mathcal{F}_f)$ to be constant means that the irreducible components of the fibers of L_f are the left cosets of the same subgroup H_Λ for one fixed primitive lattice Λ of \mathbb{Z}^n .

Remark 2. If $f \in \mathbb{C}[x_0, \dots, x_n]$ is a homogeneous polynomial, the linear system $\langle x_0 f_{x_0}, \dots, x_n f_{x_n} \rangle$ defines a rational map – still denoted by $-L_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$. It follows from Proposition 2 that, under the hypothesis of vanishing logarithmic Hessian, there are linear vector fields $X_i \in H^0(\mathbb{P}^n, T\mathbb{P}^n)$, $i = 1, \dots, k$, in the form

$$X_i = \sum_{j=0}^n \lambda_{ij} x_j \frac{\partial}{\partial x_j} \ ; \ \lambda_{ij} \in \mathbb{C}$$

which are tangent to the fibers of L_f . That is, the fibers of L_f are tangents to a Lie subalgebra of $H^0(\mathbb{P}^n, T\mathbb{P}^n)$. For the classical polar map $\nabla_f : \mathbb{P}^n \rightarrow \mathbb{P}^n$ defined just by the derivatives $\langle f_{x_0}, \dots, f_{x_n} \rangle$ it is known that the closure of a generic fiber is a union of linear spaces (see [14, Proposition 4.9]).

Some results have been obtained also to dominant maps. In [6] was obtained a classification of the homogeneous polynomials in 3 variables on which L_f is birational. This is a variant of the classification obtained in [3] of the homogeneous polynomials in 3 variables on which ∇_f is birational.

4. HESSE'S PROBLEM ON THE TORUS

Given $A = (a_{ij}) \in M(n, \mathbb{Z})$ a matrix with integer coefficients, it defines an endomorphism $\xi_A \in \text{End}(\mathbb{C}^*)^n$

$$\xi_A(x_1, \dots, x_n) = (x_1^{a_{11}} \cdots x_n^{a_{1n}}, \dots, x_1^{a_{n1}} \cdots x_n^{a_{nn}})$$

which induces an element $\xi_A^* \in \text{End}(S_n)$, where $\text{End}(S_n)$ denotes the set of endomorphisms of the coordinate ring of the torus $(\mathbb{C}^*)^n$. We note that $\xi_{AB} = \xi_B \circ \xi_A$, making it clear that, if $\det(A) = \pm 1$, then ξ_A is an automorphism with inverse $\xi_{A^{-1}}$. Any automorphism ξ of $(\mathbb{C}^*)^n$ is of the form $\xi = \xi_A$ for some invertible matrix $A \in M(n, \mathbb{Z})$ and it is classically called a *monoidal transformation*.

Theorem 1. *Let $f \in S_n$ be a Laurent polynomial. If f has vanishing logarithmic Hessian of rank $n - k$, $1 \leq k \leq n$, then there is an automorphism ξ of $(\mathbb{C}^*)^n$ such that $\xi^* f \in S_{n-k}$.*

Proof. We consider the holomorphic map $g = f \circ \mathcal{E}$, and let ∇_g its affine polar map. It follows from Proposition 2 that the irreducible components of the fibers of ∇_g are affine linear spaces $F_M + y_0 = \{y + y_0 \mid y \in F_M\}$, $y_0 \in \mathbb{C}^n$, that is, the left cosets of $F_M = M \otimes \mathbb{C}$ in \mathbb{C}^n . Where $M = \Lambda^\perp$ is a primitive lattice of \mathbb{Z}^n of rank k .

Let $\{Y_1, \dots, Y_n\}$ be a basis of \mathbb{Z}^n such that Y_1, \dots, Y_{n-k} is a basis of Λ and Y_{n-k+1}, \dots, Y_n is a basis of M . The matrix $A \in M(n, \mathbb{Z})$ in which $A(e_i) = Y_i$, $i = 1, \dots, n$, defines a linear map $A : \mathbb{C}^n \rightarrow \mathbb{C}^n$ such that $\det(A) = \pm 1$.

We have a commutative diagram

$$\begin{array}{ccccc} \mathbb{C}^n & \xrightarrow{A} & \mathbb{C}^n & & \\ \mathcal{E} \downarrow & & \mathcal{E} \downarrow & \searrow g & \\ (\mathbb{C}^*)^n & \xrightarrow{\xi_A} & (\mathbb{C}^*)^n & \xrightarrow{f} & \mathbb{C}. \end{array}$$

If $\tilde{g} = g \circ A$, then the affine polar map $\nabla_{\tilde{g}}$ is independent of the variables y_{n-k+1}, \dots, y_n , that is,

$$(4) \quad \tilde{g}_{y_1}, \dots, \tilde{g}_{y_n} \in S_{n-k}.$$

We set $\tilde{f} = \xi_A^* \cdot f \in S_n$ and assume

$$\tilde{f} = \sum_I a_I \cdot x_1^{i_1} \cdots x_n^{i_n}, \quad a_I \neq 0,$$

with $I = (i_1, \dots, i_n) \in \mathbb{Z}^n$. Hence

$$\tilde{g} = \sum_I a_I \cdot e^{i_1 y_1 + \dots + i_n y_n}, \quad a_I \neq 0.$$

Now we take the derivatives

$$\tilde{g}_{y_j} = \sum_I a_I \cdot i_j \cdot e^{i_1 y_1 + \dots + i_n y_n},$$

for $j = n - k + 1, \dots, n$. Fix $j \in \{n - k + 1, \dots, n\}$. By (4), \tilde{g}_{y_j} is independent of y_j , therefore $i_j = 0$ for all index I (consequently $\tilde{g}_{y_j} = 0$). This shows that

$$\tilde{f} = \sum_I a_I \cdot x_1^{i_1} \cdots x_{n-k}^{i_{n-k}} \in S_{n-k}.$$

□

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DEPARTAMENTO DE ANÁLISE – IM, UFF, MÁRIO SANTOS BRAGA S/N – NITERÓI, 24.020-140
 RJ BRASIL
E-mail address: `thiago@impa.br`